Orbifold Morse complex and Lagrangian Floer homology for toric orbifolds

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Outline

1. Orbifolds and Morse-Smale-Witten complex
   - Orbifolds
   - Morse-Smale-Witten complex on orbifolds

2. Lagrangian Floer homology for toric orbifolds
   - Symplectic toric orbifolds
   - Orbifold Lagrangian Floer homology
   - Orbifold holomorphic discs
   - Ingredients for computations
   - Classification of orbifold holomorphic discs
Orbifold $X$

- Locally a quotient space of $\mathbb{R}^n$ by an effective finite group action $G$. 

Example: Global quotient: $M/G$.

Example: Tear drop

Let $X$ be a quotient space (orbit space) of $X$.

Morse function on orbifold: invariant Morse function on uniformizing covers.

Objective: Find a Morse-Smale-Witten complex of $X$, computing homology $H(X)$. 

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  Then, $G$-invariant part of Morse complex of $M$ does NOT produce the homology of $M/G$. 

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When passing through non-orientable critical point, the topology does not change (of the sublevel set of quotient space)
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Then the $G$-invariant part of the Morse complex of $M$ is the Morse complex for $M/G$, whose homology is $H_*(M/G)$.
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![Diagram](image-url)
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Lifts of gradient flows can be paired in several ways.
Let $f$ be a Morse-Smale function on orbifold $X$. 

Theorem

We have $\partial^2 = 0$.

If $f$ is self-indexing, then its homology is isomorphic to $H_\bullet(X, \mathbb{R})$. 

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Lagrangian Floer homology for Toric orbifolds 
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  - smooth: normal vectors of facets meeting at a vertex form $\mathbb{Z}$-basis.
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- There exist an associated stacky fan (Borisov-Chen-Smith),
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- Studies intersection properties of Lagrangian submanifold $L_0$ and $L_1$. 

Due to Floer, Oh, Fukaya-Oh-Ohta-Ono

Chain complex is generated by intersection points $L_0 \cap L_1$

Differential is given by counting $J$-holomorphic strips connecting intersection points.

For Hamiltonian isotopy $\phi_H$, $HF(L_0, \phi_H(L_1))$ is independent of $H$.

On $S^2$, equator $L$ has $HF(L_0, \phi_L(L_1)) \sim = H^*(L)$. 

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Smooth Lagrangian Floer homology on orbifolds

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Example: Teardrop Orbifold

- There exist a smooth Floer theory, just considering smooth holomorphic maps.
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Orbi-curves

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They deform the smooth Floer theory by bulk deformation of twisted sectors.
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To compute Bott-Morse version (FOOO) of $HF(L, L)$, we need to find all such (orbifold) holomorphic discs with boundary on $L$. 

Proof consists of three ingredients:

1. Maslov index formula for holomorphic discs
2. Classification of holomorphic discs
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$$\mu_{CW} = \mu^{de} + \sum 2\iota(g_i)$$
Maslov index formula

**Theorem**

For a toric orbifold $X$ corresponding to $(\Sigma, b, P)$, let $L$ be a Lagrangian $T^n$ orbit and let $(D, (z_1, \ldots, z_k))$ be an orbi-disc with $\mathbb{Z}/m_i\mathbb{Z}$ singularity at $z_i$. Consider a holomorphic orbi-disc $w : (D, \partial D) \to (X, L)$ intersecting at each marked point $z_i$, divisor $X(v_j)$ with multiplicity $m_{i,j}$. Then the desingularized Maslov index of $w$ is given as

$$2 \sum_i \sum_j ([m_{i,j}/m_i]).$$
Using Maslov index formula, orbifold holomorphic discs can be classified.

**Theorem**

1. The holomorphic orbi-discs with one orbifold point with desingularized Maslov index 0, correspond to the twisted sectors of the toric variety.

2. The smooth Maslov index two holomorphic discs are in one to one correspondence with the vectors $b_j$'s in the stacky fan $(\Sigma, b, P)$.

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- Example of teardrop.
- Example of polytope with labels $\geq 2$. 