Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture
—A joint work with Xiangyu Cao

Zhi Lü

School of Mathematical Science
Fudan University, Shanghai

—EACAT4 in Tokyo, December 8, 2011
§1 Background—A triangle

Algebra
- Stanley-Reisner face rings

Combinatorics
- Abstract simplicial complexes

Topology
- Moment-angle complexes

Zhi Lü
Moebius transform, moment-angle complexes and Halperin-Carlsson conjecture

Some references
For the edge $a$, see

For other two edges $b, c$, see
References

For the edge $a$, see

For other two edges $b, c$, see
Let $[m] = \{1, \ldots, m\}$.

**Abstract simplicial complexes on $[m]$**

- **An abstract simplicial complex $K$ on $[m]$** is a collection of some subsets in $[m]$ such that for each $a \in K$, any subset (including $\emptyset$) of $a$ belongs to $K$.

- Each $a$ in $K$ is called a simplex of $\dim a = |a| - 1$, and $\dim K = \max_{a \in K} \{\dim a\}$. 
Let $[m] = \{1, \ldots, m\}$.

**Abstract simplicial complexes on $[m]$**

- **An abstract simplicial complex** $K$ on $[m]$ is a collection of some subsets in $[m]$ such that for each $a \in K$, any subset (including $\emptyset$) of $a$ belongs to $K$.

- Each $a$ in $K$ is called a simplex of $\dim a = |a| - 1$, and $\dim K = \max_{a \in K} \{\dim a\}$. 
Notion—Stanley-Reisner face ring

\( K \): an abstract simplicial complex on \([m]\)
\( k \): a field.

**Stanley-Reisner face ring**

\[
k(K) = k[v_1, \ldots, v_m]/I_K
\]

is called the **Stanley-Reisner face ring** of \( K \), and \( I_K \) is the ideal generated by all square-free monomials \( v_{i_1} \cdots v_{i_s} \) with \( \sigma = \{i_1, \ldots, i_s\} \notin K \).

RK: write \( k[v] = k[v_1, \ldots, v_m] \).
It is well-known that $k(K)$ is a finitely generated $\mathbb{N}^m$-graded $k[v]$-module and it has an minimal free resolution

$$0 \leftarrow k(K) \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{h-1} \leftarrow F_h \leftarrow 0 \quad (1)$$

Write $F_i = \bigoplus_{a \in \mathbb{N}^m} \left( k[v](-a) \oplus \cdots \oplus k[v](-a) \right)$ where $k[v](-a)$ is the ideal $\langle v^a \rangle$, and $v^a = v_1^{a_1} \cdots v_m^{a_m}$ for $a = (a_1, ..., a_m) \in \mathbb{N}^m$.

**Betti number**

$\beta_{i,a}^{k(K)} \in \mathbb{N}$ is called the $(i,a)$-th Betti number of $k(K)$. 
Applying the functor $\otimes_{k[v]}k$ to the sequence (1) above, one may obtain the following chain complex of $\mathbb{N}^m$-graded $k[v]$-modules:

$$0 \leftarrow F_0 \otimes_{k[v]} k \leftarrow F_1 \otimes_{k[v]} k \leftarrow \cdots \leftarrow F_h \otimes_{k[v]} k \leftarrow 0.$$ 

Define $\text{Tor}^k_{i} (k(K), k) := \frac{\ker \phi'_i}{\text{Im} \phi'_{i+1}} = F_i \otimes_{k[v]} k$ so

$$\dim_k \text{Tor}^k_{i} (k(K), k) = \text{rank} F_i = \sum_{a \in \mathbb{N}^m} \beta^{k(K)}_{i,a}.$$
A remark

It is well-known that if $a \in \mathbb{N}^m$ is not a vector in $\{0, 1\}^m$, then $\text{Tor}_i^{k[v]}(k(K), k)_a = 0$, so $\beta_{i,a}^{k(K)} = 0$.

$$\{0, 1\}^m \longleftrightarrow 2^m$$

$$\downarrow$$

write

$$\beta_{i,a}^{k(K)} := \beta_{i,a}^{k(K)}$$

where $2^m \ni a \longleftrightarrow a \in \{0, 1\}^m$. 
A general construction

\( K: \) a simplicial complex on vertex set \([m] = \{1, \ldots, m\}\)

\((X, W):\) a pair of top. spaces with \(W \subset X\).

\[ K(X, W) := \bigcup_{\sigma \in K} \left( \prod_{i \in \sigma} X \times \prod_{i \not\in \sigma} W \right) \subseteq X^m. \]

- \( \mathcal{Z}_K := K(D^2, S^1) \subset (D^2)^m \) is called the moment-angle complex on \( K \).

- \( \mathbb{R}\mathcal{Z}_K := K(D^1, S^0) \subset (D^1)^m \) is called the real moment-angle complex on \( K \).
A general construction

$K$: a simplicial complex on vertex set $[m] = \{1, ..., m\}$

$(X, W)$: a pair of top. spaces with $W \subset X$.

$$K(X, W) := \bigcup_{\sigma \in K} \left( \prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W \right) \subseteq X^m.$$  

- $\mathcal{Z}_K := K(D^2, S^1) \subset (D^2)^m$ is called the moment-angle complex on $K$.

- $\mathbb{R}\mathcal{Z}_K := K(D^1, S^0) \subset (D^1)^m$ is called the real moment-angle complex on $K$. 

Moment-angle complex
A canonical action on $\mathcal{Z}_K$

$D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and $S^1 = \partial D^2$. Since $(D^2)^m \subset \mathbb{C}^m$ is invariant under the standard action of $T^m$ on $\mathbb{C}^m$ given by

$$((g_1, \ldots, g_m), (z_1, \ldots, z_m)) \mapsto (g_1z_1, \ldots, g_mz_m),$$

$(D^2)^m$ admits a natural $T^m$-action whose orbit space is the unit cube $I^m \subset \mathbb{R}_{\geq 0}^m$. The action $T^m \rhd (D^2)^m$ then induces a canonical $T^m$-action $\Phi$ on $\mathcal{Z}_K$.

Similarly

A canonical action on $\mathbb{R}\mathcal{Z}_K$

$\mathbb{R}\mathcal{Z}_K$ admits a canonical $(\mathbb{Z}_2)^m$-action $\Phi_\mathbb{R}$ on $\mathbb{R}\mathcal{Z}_K$
On the edge $a$ of the triangle, there is the following essential result:

Hochster Theorem

For each $a \in 2^m$,

$$\tilde{H}^{|a|-i-1}(K_{|a}; k) \cong \text{Tor}_{i}^{k[v]}(k(K), k)_{a}$$

where $K_{|a} = \{\sigma \in K | \sigma \subseteq a\}$. 
On the edge $c$ of the triangle, there is the following essential result:

**Buchstaber-Panov Theorem**

As $k$-algebras,

$$H^*(Z_K; k) \cong \text{Tor}^{k[v]}(k(K), k)$$

where $k(K) = k[v]/I_K = k[v_1, \ldots, v_m]/I_K$ with $\text{deg} v_i = 2$, and $k$ is a field.
Further development—A viewpoint of analysis

- Let \( 2^*[m] = \{ f | f : 2^m \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \} \). \( 2^*[m] \) forms an algebra over \( \mathbb{Z}/2\mathbb{Z} \) in the usual way, and it has a natural basis \( \{ \delta_a | a \in 2^m \} \) where \( \delta_a \) is defined as follows:
  \[
  \delta_a(b) = 1 \iff b = a.
  \]
- Given a \( f \in 2^*[m] \), set \( \text{supp}(f) := f^{-1}(1) \)
- \( f \) is said to be nice if \( \text{supp}(f) \) is an abstract simplicial complex.

A one-one correspondence

\[
\{ \text{all nice functions in } 2^*[m] \} \leftrightarrow \{ \text{all abst. sim. subcpxes in } 2^m \}.
\]
§3.1 An algebra-combinatorics formula

The Möbius transform

On $2^m$, define a $\mathbb{Z}/2\mathbb{Z}$-valued Möbius transform

$$\mathcal{M} : 2^m \to 2^m$$

by the following way: for any $f \in 2^m$ and $a \in 2^m$,

$$\mathcal{M}(f)(a) = \sum_{b \subseteq a} f(b)$$
The following result indicates an essential relationship between \( \mathcal{M}(f) \) and the Betti numbers of \( k(K_f) \).

**Algebra–combinatorics formula (Cao-Lü)**

Suppose that \( f \in 2^m \) is nice such that \( K_f = \text{supp}(f) \) is an abstract simplicial complex on \( [m] \). Then

\[
\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^m} \beta_{i,a}^k(K_f) \delta_a
\]

where \( h \) denotes the length of the minimal free resolution of \( k(K_f) \), and \( \beta_{i,a}^k(K_f) \)'s denote the Betti numbers of \( k(K_f) \).
An algebra-combinatorics formula

Corollary

\[ |\text{supp}(\mathcal{M}(f))| \leq \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)}. \]

Proof.
\[ \mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)} \delta_a = \sum_{a \in 2^{[m]}} \left( \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \right) \delta_a \]
implies for any \( a \in \text{supp}(\mathcal{M}(f)) \), \( \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \) must be odd so
\( \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \geq 1 \). Therefore

\[ \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{k(K_f)} \geq \sum_{a \in \text{supp}(\mathcal{M}(f))} \sum_{i=0}^{h} \beta_{i,a}^{k(K_f)} \geq \sum_{a \in \text{supp}(\mathcal{M}(f))} 1 = |\text{supp}(\mathcal{M}(f))|. \]
Generalized moment-angle complex

Given an abstract simplicial complex $K$ on $[m]$, let $(X, W) = \{(X_i, W_i)\}_{i=1}^m$ be $m$ pairs of CW-complexes with $W_i \subset X_i$. Then the generalized moment-angle complex is defined as follows:

$$K(X, W) = \bigcup_{\sigma \in K} B_\sigma(X, W) \subset \prod_{i=1}^m X_i$$

where $B_\sigma(X, W) = \prod_{i=1}^m H_i$ and $H_i = \begin{cases} X_i & \text{if } i \in \sigma \\ W_i & \text{if } i \in [m] \setminus \sigma. \end{cases}$
A class of generalized moment-angle complexes

Take \((X, W) = (\mathbb{D}, S) = \{(D_i, S_i)\}_{i=1}^m\) with each CW-complex pair \((D_i, S_i)\) subject to the following conditions:

1. \(D_i\) is acyclic, that is, \(\tilde{H}_j(D_i) = 0\) for any \(j\).

2. There exists a unique \(\kappa_i\) such that \(\tilde{H}_{\kappa_i}(S_i) = \mathbb{Z}\) and \(\tilde{H}_j(S_i) = 0\) for any \(j \neq \kappa_i\).

Then our objective is to calculate the cohomology of

\[
\mathcal{Z}_K^{(\mathbb{D}, S)} := K(\mathbb{D}, S) = \bigcup_{\sigma \in K} B_\sigma(\mathbb{D}, S) \subset \prod_{i=1}^m D_i.
\]
Theorem (Cao-Lü)

As graded $k$-modules,

$$H^*(\mathcal{Z}_K^{(\mathbb{D},S)}; k) \cong \text{Tor}^k[v](k(K), k).$$

Corollary

$$\sum_i \dim_k H^i(\mathcal{Z}_K^{(\mathbb{D},S)}; k) = \sum_{i=0}^{h} \sum_{a \in 2^m} \beta_{i,a}^k(K).$$
Halperin-Carlsson conjecture

If a finite-dimensional paracompact Hausdorff space $X$ admits a free action of a torus $T^r$ (resp. a $p$-torus $(\mathbb{Z}_p)^r$, $p$ prime) of rank $r$, then the total dimension of its cohomology,

$$\sum_i \dim_k H^i(X; k) \geq 2^r$$

where $k$ is a field of characteristic 0 (resp. $p$).
Remark

- Historically, the above conjecture in the $p$-torus case originates from the work of P. A. Smith in 1950s.
- For the case of a $p$-torus $(\mathbb{Z}_p)^r$ freely acting on a finite CW-complex homotopic to $(S^n)^k$ suggested by P. E. Conner, the problem has made an essential progress.
- In the general case, the inequality was conjectured by S. Halperin for the torus case, and by G. Carlsson for the $p$-torus case.
- So far, the conjecture holds if $r \leq 3$ in the torus and 2-torus cases and if $r \leq 2$ in the odd $p$-torus case. Also, many authors have given contributions to the conjecture in many different aspects.
Recall that

\[ \sum_i \dim_k H^i(\mathcal{Z}^{(\mathbb{D},S)}_{K_f}; k) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{K_f} \geq |\text{supp}(\mathcal{M}(f))|. \]

We can upbuild a method of compressing $\text{supp}(f)$ to get the desired lower bound of $|\text{supp}(\mathcal{M}(f))|$. 

**Theorem (Cao-Lü)**

For any nice $f \in 2^{[m]^{**}}$, there exists some $a \in \text{supp}(f)$ such that

\[ |\text{supp}(\mathcal{M}(f))| \geq 2^m - |a|. \]
Application to free actions

Theorem (Cao-Lü)

Let \( H \) (resp. \( H_\mathbb{R} \)) be a rank \( r \) subtorus of \( T^m \) (resp. \( (\mathbb{Z}_2)^m \)). If \( H \) (resp. \( H_\mathbb{R} \)) can act freely on \( \mathcal{Z}_K \) (resp. \( \mathbb{R}\mathcal{Z}_K \)), then

\[
\sum_i \dim_k H_i(\mathcal{Z}_K; k) = \sum_i \dim_k H_i(\mathbb{R}\mathcal{Z}_K; k) \geq 2^r.
\]

Remark

The action of \( H \) (resp. \( H_\mathbb{R} \)) on \( \mathcal{Z}_K \) (resp. \( \mathbb{R}\mathcal{Z}_K \)) is naturally regarded as the restriction of the \( T^m \)-action \( \Phi \) to \( H \) (resp. the \( (\mathbb{Z}_2)^m \)-action \( \Phi_\mathbb{R} \) to \( H_\mathbb{R} \)).
Corollary

The Halperin–Carlsson conjecture holds for $\mathbb{Z}_K$ (resp. $\mathbb{R}\mathbb{Z}_K$) under the restriction of the $T^m$-action $\Phi$ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_R$).

Remark

Using a different method, Yury Ustinovsky has also recently proved the Halperin’s toral rank conjecture for the moment-angle complexes with the restriction of natural tori actions.
Recent development

- Li Yu: the case of $\mathbb{Z}_2^m \acts M$ such that $M/\mathbb{Z}_2^m$ is a small cover.
- Y. Kamishima and M. Nakayama: the case of aspherical flat manifold with free $\mathbb{Z}_2^m$-action