ON THE HIT PROBLEM
FOR THE POLYNOMIAL ALGEBRA

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Tokyo - 07 December 2011
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References
1. The hit problem for the polynomial algebra

Let $V_k$ be an elementary abelian 2-group of rank $k$. Denote by $BV_k$ the classifying space of $V_k$. Then $P_k := H^* (BV_k) \cong F_2 [x_1, x_2, \ldots, x_k]$, a polynomial algebra on $k$ generators $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $F_2$ of two elements. Being the cohomology of a space, $P_k$ is a module over the mod 2 Steenrod algebra $A$. The action of $A$ on $P_k$ can explicitly be given by the formula

$$Sq^i (x_j) = \begin{cases} x_j, & i = 0 \\ x_j^2, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

and subject to the Cartan formula $Sq^i (fg) = \sum_{j=0}^{n} Sq^i (f) Sq^{n-j} (g)$, for $f, g \in P_k$. 

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1. The hit problem for the polynomial algebra

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  x_j, & i = 0, \\
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and subject to the Cartan formula

$$Sq^n(fg) = \sum_{i=0}^{n} Sq^i(f) Sq^{n-i}(g),$$

for $f, g \in P_k$. 
1. The hit problem for the polynomial algebra

A polynomial \( f \) in \( P_k \) is called \textit{hit} if it can be written as a finite sum 
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  f = \sum_{i>0} Sq^i(f_i) 
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for some polynomials \( f_i \).

That means \( f \) belongs to \( \mathcal{A}^+P_k \), where \( \mathcal{A}^+ \) denotes the augmentation ideal in \( \mathcal{A} \).
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In other words, we want to find a basis of the $\mathbb{F}_2$-vector space $QP_k := P_k/\mathcal{A}^+.P_k = \mathbb{F}_2 \otimes \mathcal{A} P_k$. 
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Let $GL_k = GL_k(\mathbb{F}_2)$ be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on $P_k$ by matrix substitution. Since the two actions of $\mathcal{A}$ and $GL_k$ upon $P_k$ commute with each other, there is an action of $GL_k$ on $QP_k$. 
1. The hit problem for the polynomial algebra

The subspace of degree $n$ homogeneous polynomials $(P_k)_n$ and its quotient $(QP_k)_n$ are $GL_k$-subspaces of the spaces $P_k$ and $QP_k$ respectively.
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The hit problem was first studied by Peterson [4], Wood [10], Singer [7], and Priddy [5], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups.
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Several aspects of the hit problem were then investigated by Boardman, Bruner, Hà, Hưng, Carlisle, Crabb, Hubbuck, Giambalvo, Nam, Janfada, Kameko, Minami, Repka, Selick, Silverman, Walker, Wood and others.
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The vector space \(QP_k\) was explicitly calculated by Peterson [4] for \(k = 1, 2\), by Kameko [2] for \(k = 3\). The case \(k = 4\) has been treated by Kameko. However the manuscript unpublished at the time.
2. Kameko’s conjecture

The following is the early results on the hit problem.

Theorem (Peterson (Abs. AMS. 87 [4]), Kameko (PhD Thesis 90 [2]))

For every nonnegative integer \( n \),

1. \( \dim (QP_1^n) \leq 1 \),
2. \( \dim (QP_2^n) \leq 3 \),
3. \( \dim (QP_3^n) \leq 21 \).

Carlisle and Wood showed that the dimension of the vector space \( (QP_k^n) \) is uniformly bounded by a number depended only on \( k \).

In 1990, Kameko made the following conjecture in his Johns Hopkins University Ph. D. thesis [2].

Conjecture (Kameko (Ph.D. Thesis 90 [2])). For every nonnegative integer \( n \),

\[
\dim (QP_k^n) \leq \prod_{1 \leq i \leq k} (2i - 1)
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The above theorem implies that this conjecture is true for \( k \leq 3 \).
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**Conjecture (Kameko (Ph.D. Thesis 90 [2])).** For every nonnegative integer $n$,  
$$\dim(QP_k)_n \leq \prod_{1 \leq i \leq k} (2^i - 1).$$

The above theorem implies that this conjecture is true for $k \leq 3$. 
3. Kameko’s squaring operation

3.1. The $\mu$-function.

The $\mu$-function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer $n$, $\mu(n) = \min\{r \in \mathbb{N} : n = \sum_{1 \leq i \leq r} (2^{d_i} - 1)\}$, where $d_i > 0$.

Theorem

For every positive integer $n$, $\mu(n) = s$ if and only if there exist integers $d_1 > d_2 > \ldots > d_{s-1} \geq d_s > 0$ such that

$$n = 2^{d_1} + 2^{d_2} + \ldots + 2^{d_{s-1}} + 2^{d_s - s}.$$
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If $\mu(n) > k$, then $(QP_k)_n = 0$. 

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3.2. Squaring operation.

One of the main tools in the study of the hit problem is the dual of Kameko’s squaring $Sq^0_* : (QP_k)^{GL_k} \rightarrow (QP_k)^{GL_k}$. This homomorphism is induced by the $GL_k$-homomorphism $\widetilde{Sq^0}_* : QP_k \rightarrow QP_k$. 
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$$\widetilde{Sq}_*(x) = \begin{cases} y, & \text{if } x = x_1x_2 \ldots x_ky^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in P_k$. 
3. Kameko’s squaring operation

Note that $\tilde{Sq}_0^*$ is not an $A$-homomorphism. However, $\tilde{Sq}_0^* Sq^{2t} = Sq^t \tilde{Sq}_0^*$, for any nonnegative integer $t$.

**Theorem (Kameko (PhD Thesis 90 [2]))**

Let $m$ be a positive integer. If $\mu(2m + k) = k$, then 
$$(\tilde{Sq}_0^*)_m : (QP_k)_{2m+k} \to (QP_k)_m$$
is an isomorphism of $GL_k$-modules.
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Based on the above results, the hit problem is reduced to the case of degree $n$ with $\mu(n) < k$. 
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Note that \( \tilde{Sq}_0 \) is not an \( \mathcal{A} \)-homomorphism. However, \( \tilde{Sq}_0 \cdot Sq^{2t} = Sq^t \tilde{Sq}_0 \), for any nonnegative integer \( t \).

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Let \( m \) be a positive integer. If \( \mu(2m + k) = k \), then \( (\tilde{Sq}_*)_m : (QP_k)_{2m+k} \to (QP_k)_m \) is an isomorphism of \( GL_k \)-modules.

Based on the above results, the hit problem is reduced to the case of degree \( n \) with \( \mu(n) < k \).
That means that \( n \) is of the form (1) with \( s < k \).
The hit problem in the case of degree \( n \) of the form (1) with \( s = k - 1 \), \( d_{i-1} - d_i > 1 \) for \( 2 \leq i < k \) and \( d_{k-1} > 1 \) was studied by Crabb-Hubbuck [1], Nam [3] and Repka-Selick [6].
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In the next part, we present some results on the hit problem for the cases of degree $n$ of the form (1) for either $s = k - 1$ or $s = k - 2$.  

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3. Some results on the hit problem in generic degree

We study the hit problem in the case of degree $n$ of the form (1) with $s = k - 1$ and $d_k - 2 \geq d_k - 1 \geq k - 1$. In this case, we have

Theorem

Let $n = \sum_{1 \leq i \leq k-1} (2d_i - 1)$ with $d_i$ positive integers such that $d_i - 1 - d_i \geq i - 1$, $3 \leq i \leq k - 1$, and $d_{k-1} \geq k - 1$.

(1) If $d_1 - d_2 \geq 2$, then $\dim(QP_k n) = \prod_{1 \leq i \leq k} (2i - 1)$.

(2) If $d_1 - d_2 = 1$, then $\dim(QP_k n) = 2 \prod_{3 \leq i \leq k-1} (2i - 1)$.

For the case $d_i - 1 - d_i \geq i$, $3 \leq i \leq k - 1$, and $d_{k-1} \geq k - 1$, these results are due to Nam (Adv. Math. 2004 [3]). By this theorem, Kameko’s conjecture is true in generic degree.
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We study the hit problem in the case of degree $n$ of the form (1) with $s = k - 1$ and $d_{k-2} > d_{k-1} \geq k - 1$. In this case, we have
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Let $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$ with $d_i$ positive integers such that 
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1. If $d_1 - d_2 \geq 2$, then $\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1)$.
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1. If $d_1 - d_2 \geq 2$, then $\dim(\mathcal{QP}_k)_n = \prod_{1 \leq i \leq k} (2^i - 1)$.
2. If $d_1 - d_2 = 1$, then $\dim(\mathcal{QP}_k)_n = 2 \prod_{3 \leq i \leq k} (2^i - 1)$.

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By this theorem, Kameko’s conjecture is true in generic degree.
Part II. The negative answer to Kameko’s conjecture

1. Part I. The hit problem for the polynomial algebra

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3. Part III. The case $k = 4$

4. References
1. Main Theorem

In this part, we present some results on the hit problem for the case of degree \(n\) of the form (1) with \(s = k - 1\) and \(d_{k-2} \geq d_{k-1} \geq k - 1\).

The following theorem gives an inductive formula for the dimension of \((QP^k)^n\) in this case.

**Theorem (Main Theorem)**

Let \(n = \sum_{1 \leq i \leq k-1} (2d_i - 1)\) with \(d_i\) positive integers such that \(d_1 > d_2 > \ldots > d_{k-2} \geq d_{k-1}\), and let \(m = \sum_{1 \leq i \leq k-2} (2d_i - d_{k-1} - 1)\). If \(d_{k-1} \geq k - 1 \geq 1\), then

\[
\dim (QP^k)^n = (2k - 1) \dim (QP^{k-1})^m.
\]

For \(d_{k-1} = k - 1\), this theorem is new.

However, for \(d_{k-1} = k - 1\), the theorem follows from the results in Nam (Adv. Math. 2004 [3]) and S (Adv. Math. 2010 [9]).
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$$\dim(QP_k)_n = (2^k - 1) \dim(QP_{k-1})_m.$$  

For $d_{k-1} \geq k$, the theorem follows from the results in Nam (Adv. Math. 2004 [3]) and S (Adv. Math. 2010 [9]). However, for $d_{k-1} = k - 1$, this theorem is new.
2. The negative answer to Kameko’s conjecture

By induction on $k$, using Main Theorem for $d_{k-2} = d_k - 1 \geq k - 1$ and the fact that the dual of Kameko’s squaring operation is an epimorphism, one get the following.

**Theorem**

Let $n = \sum_{1 \leq i \leq k-2} (2d_i - 1)$ with $d_i$ positive integers and let $d_{k-1} = 1$, $n_r = \sum_{1 \leq i \leq r-2} (2d_i - 1)$ with $r = 5, 6, \ldots, k$. If $d_1 - d_2 \geq 4$, $d_i - 2 - d_{i-1} \geq i$, for $4 \leq i \leq k$ and $k \geq 5$, then $\dim(\text{QP}_k^n) = \prod_{1 \leq i \leq k} (2i-1) + \sum_{5 \leq r \leq k} (\prod_{r+1 \leq i \leq k} (2i-1)) \dim(\text{Ker}(\tilde{\text{Sq}}_{0^*}))_{n_r}$, where $(\tilde{\text{Sq}}_{0^*})_{n_r}$ denotes Kameko’s squaring $\tilde{\text{Sq}}_0$ in degree $2n_r + r$. Here, by convention, $\prod_{r+1 \leq i \leq k} (2i-1) = 1$ for $r = k$. 
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Let \( n = \sum_{1 \leq i \leq k-2} (2^{d_i} - 1) \) with \( d_i \) positive integers and let \( d_{k-1} = 1, \ n_r = \sum_{1 \leq i \leq r-2} (2^{d_i} - d_{r-1} - 1) - 1 \) with \( r = 5, 6, \ldots, k \). If \( d_1 - d_2 \geq 4, \ d_i - 2 - d_{i-1} \geq i, \) for \( 4 \leq i \leq k \) and \( k \geq 5 \), then

\[
\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1) + \sum \left( \prod_{5 \leq r \leq k} (2^i - 1) \right) \dim \ker(\widetilde{Sq}_0^*)_{n_r},
\]

where \( (\widetilde{Sq}_0^*)_{n_r} : (QP_r)_{2n_r + r} \to (QP_r)_{n_r} \) denotes Kameko’s squaring \( \widetilde{Sq}_* \) in degree \( 2n_r + r \). Here, by convention, \( \prod_{r+1 \leq i \leq k} (2^i - 1) = 1 \) for \( r = k \).
2. The negative answer to Kameko’s conjecture

In order to conclude that Kameko’s conjecture is false in degree $2n_k + k$ for any $k \geq 5$, it suffices to show that $\text{Ker}(\widetilde{Sq}_0)^r$ is nonzero.
2. The negative answer to Kameko’s conjecture

In order to conclude that Kameko’s conjecture is false in degree $2n_k + k$ for any $k \geq 5$, it suffices to show that $\text{Ker}(\tilde{Sq}^0_{\ast})_{nr}$ is nonzero.

Obviously $2n_r + r = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_{r-2}} - r + 2$, where $e_i = d_i - d_{r-1} + 1$ for $1 \leq i \leq r - 2$. 
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In order to conclude that Kameko’s conjecture is false in degree $2n_k + k$ for any $k \geq 5$, it suffices to show that $\text{Ker}(\widetilde{Sq}_0^*)_{n_r}$ is nonzero.

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Consider the element $x = x_1^{2^{e_1} - 1}x_2^{2^{e_2} - 1}\ldots x_{r-2}^{2^{e_{r-2}} - 1}$, in degree $2n_r + r$. The element is called a spike, i.e. a monomial whose exponents are all of the form $2^e - 1$ for some $e$. 
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It is well-known that the class $[x]$ in $(QP_r)_{2n_r + r}$ of a spike $x$ is nonzero.

Indeed, one has

$$Sq^a x_j^{2^e - 1 - a} = \left(\begin{array}{c} 2^e - 1 - a \\ a \end{array}\right) x_j^{2^e - 1} = 0,$$

as $\left(\begin{array}{c} 2^e - 1 - a \\ a \end{array}\right) = 0$ in $\mathbb{F}_2$ for arbitrary $j$ and any $a > 0$. Hence, a spike cannot be hit by any Steenrod operation of positive degree.
On the other hand, since the exponents of \( x_{r-1} \) and \( x_r \) in \( x \) are zero, 
\((\tilde{Sq}_*)_{n_r}([x]) = 0\). Thus, we have \( \text{Ker}(\tilde{Sq}_*)_{n_r} \neq 0 \) for \( r = 5, 6, \ldots, k \).
2. The negative answer to Kameko’s conjecture

On the other hand, since the exponents of $x_{r-1}$ and $x_r$ in $x$ are zero, $(\widetilde{Sq}_r^0)_{n_r}([x]) = 0$. Thus, we have $\text{Ker}(\widetilde{Sq}_r^0)_{n_r} \neq 0$ for $r = 5, 6, \ldots, k$. Therefore, by the above theorem, Kameko’s conjecture is not true in degree $n = 2n_k + k$ for any $k \geq 5$, where $n_k = 2^{d_1-1} + 2^{d_2-1} + \ldots + 2^{d_k-2-1} - k + 1$. So, we get
2. The negative answer to Kameko’s conjecture

On the other hand, since the exponents of $x_{r-1}$ and $x_r$ in $x$ are zero, $(\widetilde{Sq}_0^r)[x]\rangle = 0$. Thus, we have $\ker(\widetilde{Sq}_0^r)[x] \neq 0$ for $r = 5, 6, \ldots, k$. Therefore, by the above theorem, Kameko’s conjecture is not true in degree $n = 2n_k + k$ for any $k \geq 5$, where $n_k = 2^{d_1 - 1} + 2^{d_2 - 1} + \ldots + 2^{d_k - 2} - k + 1$. So, we get

**Corollary.** Kameko’s conjecture is not true for any $k > 4$. 
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On the other hand, since the exponents of $x_{r-1}$ and $x_r$ in $x$ are zero, $(\widetilde{Sq}_*)_{n_r}([x]) = 0$. Thus, we have $\text{Ker}(\widetilde{Sq}_*)_{n_r} \neq 0$ for $r = 5, 6, \ldots, k$. Therefore, by the above theorem, Kameko’s conjecture is not true in degree $n = 2n_k + k$ for any $k \geq 5$, where

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**Corollary.** Kameko’s conjecture is not true for any $k > 4$.

However, we have $\dim (QP_k)_{2n_k + k} = \dim (QP_k)_{n_k} + \dim \text{Ker}(\widetilde{Sq}_*)_{n_k}$. 

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Then, from Main Theorem we obtain

$$\dim(QP_k)_{n_k} = \prod_{1 \leq i \leq k} (2^i - 1) + \sum_{5 \leq r < k} \left( \prod_{r+1 \leq i \leq k} (2^i - 1) \right) \dim \text{Ker}(\widetilde{Sq}_*)_{n_r}.$$
2. The negative answer to Kameko’s conjecture

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**Corollary.** Kameko’s conjecture is not true for any $k > 4$.

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Hence, for $k > 5$, Kameko’s conjecture is not true in degree $n_k$. 
3. Proof of Main Theorem

We denote $N_k = \{ (i_1; I) ; I = (i_1, i_2, ..., i_r), 1 \leq i_2 < ... < i_r \leq k, 0 \leq r < k \}$.

For $1 \leq r < k$, we set $N_{r-1} \cup r = \{ (i_1; I \cup r) ; (i_1; I) \in N_{r-1} \}$. Then we have $N_k = (N_1 \cup 2) \cup ... \cup (N_{k-1} \cup k) \cup \{(1; \emptyset), ..., (k; \emptyset)\}$.

For $1 \leq i \leq k$, define the homomorphism $f_i = f_k; i : P_{k-1} \to P_k$ of algebras by substituting $f_i(x_j) = \{ x_j, \text{if } 1 \leq j < i, x_j + 1, \text{if } i \leq j < k \}$. 
We denote

\[ \mathcal{N}_k = \{ (i; l); l = (i_1, i_2, \ldots, i_r), 1 \leq i < i_1 < \ldots < i_r \leq k, \ 0 \leq r < k \}. \]
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For $1 \leq r < k$, we set $\mathcal{N}_{r-1} \cup r = \{(i; I \cup r); (i; I) \in \mathcal{N}_{r-1}\}$. Then we have

$$\mathcal{N}_k = (\mathcal{N}_1 \cup 2) \cup \ldots \cup (\mathcal{N}_{k-1} \cup k) \cup \{(1; \emptyset), \ldots, (k; \emptyset)\}.$$
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For \( 1 \leq i \leq k \), define the homomorphism \( f_i = f_{k;i} : P_{k-1} \rightarrow P_k \) of algebras by substituting

\[ f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases} \]
3. Proof of Main Theorem

Let \((i; l) \in \mathcal{N}_k, u \in \mathbb{N}\). Set \(x_{(l,u)} = x_{i_u}^{2^{r-1}+\ldots+2^{r-u}} \prod_{u < t \leq r} x_{i_t}^{2^{r-t}}\) for \(1 \leq u \leq r\), and \(x_{(l,u)} = 1\) for \(r = 0\).
Let \((i; I) \in \mathcal{N}_k, u \in \mathbb{N}\). Set \(x(I, u) = x_{i_{u}}^{2r-1} \cdots 2^{r-u} \prod_{u < t \leq r} x_{i_{t}}^{2r-t}\) for \(1 \leq u \leq r\), and \(x(I, u) = 1\) for \(r = 0\).

For a monomial \(x = x_1^{a_1} x_2^{a_2} \cdots x_{k-1}^{a_{k-1}}\) in \(P_{k-1}\), we define the monomial \(\phi(i; I)(x)\) in \(P_k\) by setting

\[
\phi(i; I)(x) = \begin{cases} 
(x_i^{2r-1} f_i(x))/x(I, u), & \text{if } a_{i_1-1} = \cdots = a_{i_{u-1}-1} = 2^r - 1, \ a_{i_u-1} > 2^r - 1, \alpha_{r-t}(a_{i_{u-1}}) = 1, \\
0, & \text{otherwise,}
\end{cases}
\]
3. Proof of Main Theorem

Let \((i; l) \in \mathcal{N}_k, u \in \mathbb{N}\). Set \(x_{(l,u)} = x_{i_u}^{2r-1} \cdots + x_{r}^{2r-u} \prod_{u < t \leq r} x_{i_t}^{2r-t}\) for \(1 \leq u \leq r\), and \(x_{(l,u)} = 1\) for \(r = 0\). For a monomial \(x = x_1^{a_1} x_2^{a_2} \cdots x_{k-1}^{a_{k-1}}\) in \(P_{k-1}\), we define the monomial \(\phi_{(i;l)}(x)\) in \(P_k\) by setting

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& a_{i_u-1} > 2r - 1, \alpha_{r-t}(a_{i_u-1}) = 1, \\
0, & 1 \leq t \leq u, \alpha_{r-t}(a_{i_t-1}) = 1, \ u < t \leq r, \\
& \text{otherwise}, 
\end{cases}
\]

Then we have an \(\mathbb{F}_2\)-linear map \(\phi_{(i;l)} : P_{k-1} \to P_k\).
3. Proof of Main Theorem

In particular, \( \phi(i;\emptyset) = f_i \).

Let \( X = x_1x_2\ldots,x_{k-1} \in P_{k-1} \). If \( a_{j-1} = 0, j = i_1, i_2, \ldots, i_{u-1} \) and \( a_{i_u-1} > 0 \), then

\[
\phi(i;I)(X^{2^r-1}x^{2^r}) = \phi(i_u;J_u)(X^{2^r-1})f_i(x)^{2^r},
\]

where \( J_u = (i_{u+1}, \ldots, i_r) \).
3. Proof of Main Theorem

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\[
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\]

where \( J_u = (i_{u+1}, \ldots, i_r) \).

Let \( B \) be a finite subset of \( P_{k-1} \) consisting of some homogeneous polynomials in degree \( n \). We set

\[
\Phi^0(B) = \bigcup_{1 \leq i \leq k} \phi(i;\emptyset)(B) = \bigcup_{1 \leq i \leq k} f_i(B).
\]
\[
\Phi^+(B) = \bigcup_{(i;l) \in \mathcal{N}_k, 0<\ell(l) \leq k-1} \phi(i;l)(B).
\]
\[
\Phi(B) = \Phi^0(B) \cup \Phi^+(B).
\]
Main Theorem follows from the following:

Proposition
Let \( n, m \) be as in Main Theorem and \( d^{k-1} \geq k-1 \). If \( B^{k-1}(n) \) is a minimal set of generators for the \( A \)-module \( P^{k-1} \) in degree \( n \), then \( B^k(n) = \Phi(B^{k-1}(n)) \) is also a minimal set of generators for the \( A \)-module \( P^k \) in degree \( n \).

For \( d^{k-1} \geq k \), this proposition is a modification of a result in Nam (Adv. Math. 2004 [3]). For \( d^{k-1} > k \) and \( d^{k-2} = d^{k-1} \), it was proved in S (Adv. Math. 2010 [9].)
3. Proof of Main Theorem

Main Theorem follows from the following:

**Proposition**

Let $n, m$ be as in Main Theorem and $d_{k-1} \geq k - 1$. If $B_{k-1}(n)$ is a minimal set of generators for $A$-module $P_{k-1}$ in degree $n$, then $B_k(n) = \Phi(B_{k-1}(n))$ is also a minimal set of generators for $A$-module $P_k$ in degree $n$. 
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Part III. The case $k = 4$
1. The case $k = 4$

For $k = 4$ the hit problem is reduced to the cases of degree $n$ with $\mu(n) < 4$. Since $\alpha(n) + \mu(n) < 4$, it suffices to consider six cases:

1. $n = 2s + 2 - 3, \mu(n) = 3, \alpha(n + 3) = 1$,

2. $n = 2s + 1 - 2, \mu(n) = 2, \alpha(n + 2) = 1$,

3. $n = 2s - 1, \mu(n) = 1, \alpha(n + 1) = 1$,

4. $n = 2s + t + 1 + 2s + t + 2 - 3, \mu(n) = 3, \alpha(n + 3) = 2$,

5. $n = 2s + t + 2s - 2, \mu(n) = 2, \alpha(n + 2) = 2$,

6. $n = 2s + t + u + 2s + t + 2 - 3, \mu(n) = 3, \alpha(n + 3) = 3$,
1. The case $k = 4$

For $k = 4$ the hit problem is reduced to the cases of degree $n$ with $\mu(n) < 4$. Since $\alpha(n + \mu(n)) \leq \mu(n) < 4$, it suffices to consider six cases:
1. The case $k = 4$

For $k = 4$ the hit problem is reduced to the cases of degree $n$ with $\mu(n) < 4$. Since $\alpha(n + \mu(n)) \leq \mu(n) < 4$, it suffices to consider six cases:

1) $n = 2^{s+2} - 3$, $\mu(n) = 3, \alpha(n + 3) = 1$,
2) $n = 2^{s+1} - 2$, $\mu(n) = 2, \alpha(n + 2) = 1$,
3) $n = 2^s - 1$, $\mu(n) = 1, \alpha(n + 1) = 1$,
4) $n = 2^{s+t+1} + 2^{s+1} - 3$, $\mu(n) = 3, \alpha(n + 3) = 2$,
5) $n = 2^{s+t} + 2^s - 2$, $\mu(n) = 2, \alpha(n + 2) = 2$,
6) $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$, $\mu(n) = 3, \alpha(n + 3) = 3$,

where $s, t, u$ are the positive integers.
2. The cases $\alpha(n + \mu(n)) = 1$

2.1. If $\mu(n) = 3$, then $n = 2s + 2 - 3 = 2s + 1 + 2s - 3$.

From Main Theorem we have $\dim(QP^4_{n}) = 15 \dim(QP^3_{1}) = 45$ for $s \geq 3$.

By a direct computation using Kameko's results for $k = 3$, we have $\dim(QP^4_{5}) = 15$ for $s = 1$, and $\dim(QP^4_{13}) = 35$ for $s = 2$.

2.2. If $\mu(n) = 2$, then $n = 2s + 1 - 2 = 2(2s - 3) + 4$. Hence, we have $QP^4_{n} \sim (QP^4_{2s - 3}) \oplus \ker(\tilde{Sq}^0_{2s - 3})$.

By a direct computation, we get $n = 2s + 1 - 2$ for $s = 1$, $s = 2$, $s = 3$, $s = 4$, and $s \geq 5$.

2.3. If $\mu(n) = 1$, then $n = 2s - 1$. By a direct computation, we get $n = 2s - 1$ for $s = 1$, $s = 2$, $s = 3$, $s = 4$, and $s \geq 5$. 

$\dim(QP^4_{n}) | 4, 14, 35, 75, 89, 85.$
2. The cases $\alpha(n + \mu(n)) = 1$

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\[
\begin{array}{c|ccc}
   n = 2^{s+2} - 3 & s = 1 & s = 2 & s \geq 3 \\
\hline
   \dim(QP_4)_n & 15 & 35 & 45 \\
\end{array}
\]
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$$
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 n = 2^{s+2} - 3 & | & s = 1 \quad s = 2 \quad s \geq 3 \\
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\end{array}
$$

2.2. If $\mu(n) = 2$, then $n = 2^{s+1} - 2 = 2(2^s - 3) + 4$. Hence, we have $(QP_4)_n \cong (QP_4)_{2^s-3} \oplus \ker(Sq^0_{2^s-3})$.

By a direct computation, we get

$$
\begin{array}{cccccc}
 n = 2^{s+1} - 2 & | & s = 1 \quad s = 2 \quad s = 3 \quad s = 4 \quad s \geq 5 \\
 \dim(QP_4)_n & | & 6 \quad 24 \quad 50 \quad 70 \quad 80 \\
\end{array}
$$
2. The cases $\alpha(n + \mu(n)) = 1$

2.1. If $\mu(n) = 3$, then $n = 2^{s+2} - 3 = 2^{s+1} + 2^s + 2^s - 3$.

From Main Theorem we have $\text{dim}(QP_4)_n = 15 \text{dim}(QP_3)_1 = 45$ for $s \geq 3$.

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$\text{dim}(QP_4)_5 = 15$ for $s = 1$, and $\text{dim}(QP_4)_{13} = 35$ for $s = 2$.

$$
\begin{array}{c|ccc}
  n = 2^{s+2} - 3 & s = 1 & s = 2 & s \geq 3 \\
  \text{dim}(QP_4)_n & 15 & 35 & 45
\end{array}
$$

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$(QP_4)_n \cong (QP_4)_{2^s - 3} \oplus \text{Ker}(\tilde{Sq}^0)_{2^s - 3}$.

By a direct computation, we get

$$
\begin{array}{c|cccccc}
  n = 2^{s+1} - 2 & s = 1 & s = 2 & s = 3 & s = 4 & s \geq 5 \\
  \text{dim}(QP_4)_n & 6 & 24 & 50 & 70 & 80
\end{array}
$$

2.3. If $\mu(n) = 1$, then $n = 2^s - 1$. By a direct computation, we get

$$
\begin{array}{c|cccccc}
  n = 2^s - 1 & s = 1 & s = 2 & s = 3 & s = 4 & s = 5 & s \geq 6 \\
  \text{dim}(QP_4)_n & 4 & 14 & 35 & 75 & 89 & 85
\end{array}
$$
3. The cases $\alpha(n + \mu(n)) = 2$

If $\mu(n) = 3$, then

$$n = 2s + t + 1 + 2s + 1 - 3 = 2s + t + 1 + 2s + 2s - 3.$$ 

By Main Theorem, $\dim(QP_4^n) = 15 \dim(QP_3^{2t+1-1})$. So, by a direct computation using Kameko's results for $k = 3$, we see that $\dim(QP_4^n)$ is given by the following table:

<table>
<thead>
<tr>
<th>$s$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t \geq 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1$</td>
<td>46, 87, 136, 150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s = 2$</td>
<td>94, 135, 180, 195</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s \geq 3$</td>
<td>105, 150, 195, 210</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. The cases $\alpha(n + \mu(n)) = 2$

3.1. If $\mu(n) = 3$, then $n = 2^{s+t+1} + 2^{s+1} - 3 = 2^{s+t+1} + 2^s + 2^s - 3$. By Main Theorem, $\dim(QP_4)_n = 15 \dim(QP_3)_{2^{t+1}-1}$. So, by a direct computation using Kameko’s results for $k = 3$, we see that $\dim(QP_4)_n$ is given by the following table:

<table>
<thead>
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<th>$t$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>46</td>
<td>94</td>
<td>105</td>
</tr>
<tr>
<td>2</td>
<td>87</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>$\geq 4$</td>
<td>136</td>
<td>180</td>
<td>195</td>
</tr>
</tbody>
</table>
3. The cases $\alpha(n + \mu(n)) = 2$

3.1. If $\mu(n) = 3$, then $n = 2^{s+t+1} + 2^{s+1} - 3 = 2^{s+t+1} + 2^s + 2^s - 3$. By Main Theorem, $\dim(QP_4)_n = 15 \dim(QP_3)_{2^{t+1}-1}$. So, by a direct computation using Kameko’s results for $k = 3$, we see that $\dim(QP_4)_n$ is given by the following table:

<table>
<thead>
<tr>
<th>$n = 2^{s+t+1} + 2^{s+1} - 3$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t \geq 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = 1$</td>
<td>46</td>
<td>87</td>
<td>136</td>
<td>150</td>
</tr>
<tr>
<td>$s = 2$</td>
<td>94</td>
<td>135</td>
<td>180</td>
<td>195</td>
</tr>
<tr>
<td>$s \geq 3$</td>
<td>105</td>
<td>150</td>
<td>195</td>
<td>210</td>
</tr>
</tbody>
</table>
3. The cases \( \alpha(n + \mu(n)) = 2 \)

3.2. If \( \mu(n) = 2 \), then \( n = 2^{s+t} + 2^s - 2 = 2m + 4 \), where \( m = 2^{s+t-1} + 2^{s-1} - 3 \). So, \( \dim(QP_4)_n \cong (QP_4)_m \oplus \text{Ker}(\tilde{Sq}^0)_m \). By a direct computation, we have
3. The cases $\alpha(n + \mu(n)) = 2$

3.2. If $\mu(n) = 2$, then $n = 2^{s+t} + 2^s - 2 = 2m + 4$, where

$m = 2^{s+t-1} + 2^{s-1} - 3$. So, $\dim(QP_4)_n \cong (QP_4)_m \oplus \text{Ker}(\widetilde{Sq}^0)_m$. By a direct computation, we have

\[
\begin{array}{c|cccccc}
 n = 2^{s+t} + 2^s - 2 & t = 1 & t = 2 & t = 3 & t = 4 & t = 5 & t \geq 6 \\
 s = 1 & 21 & 55 & 73 & 95 & 115 & 125 \\
 s = 2 & 70 & 126 & 165 & 179 & 175 & 175 \\
 s = 3 & 116 & 192 & 241 & 255 & 255 & 255 \\
 s = 4 & 164 & 240 & 285 & 300 & 300 & 300 \\
 s \geq 5 & 175 & 255 & 300 & 315 & 315 & 315 \\
\end{array}
\]
4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2s + t + u + 2s + t - 3$. By Main Theorem, for $s \geq 3$, dim$(QP^4_n) = 15 \dim(QP^3_m)$ with $m = 2t + u + 2t - 2$.

Hence, by a direct computation using Kameko's results for $k = 3$, one get $n|t = 1$ $t = 2$ $t \geq 3$ $t = 1$ $t = 1$ $t \geq 3$ $t \geq 2$ $u \geq 3$ $u \geq 2$ $s = 1$ $64 155 140 140 120 225 210$

From the above results, we have Corollary. Kameko's conjecture is true for $k = 4$. 

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4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$. 
4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$. Hence, by a direct computation using Kameko’s results for $k = 3$, one get

\[ \begin{array}{cccccc}
\hline
& s & t & u & s & t \\
\hline
1 & 1 & 2 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 & 2 & 2 \\
\hline
\end{array} \]
4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2^{s+t} + u + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$.

Hence, by a direct computation using Kameko’s results for $k = 3$, one get

<table>
<thead>
<tr>
<th>$s$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>64</td>
<td>155</td>
<td>140</td>
<td>140</td>
<td>120</td>
<td>225</td>
<td>210</td>
<td></td>
</tr>
</tbody>
</table>

Corollary. Kameko’s conjecture is true for $k = 4$. 
4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$.

Hence, by a direct computation using Kameko’s results for $k = 3$, one gets:

<table>
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<tr>
<th>$n$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t \geq 3$</th>
<th>$t = 1$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t &gt; 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u = 1$</td>
<td>$u = 1$</td>
<td>$u = 1$</td>
<td>$u = 2$</td>
<td>$u \geq 3$</td>
<td>$u \geq 2$</td>
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</tr>
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<td>155</td>
<td>140</td>
<td>140</td>
<td>120</td>
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<td>210</td>
</tr>
<tr>
<td>$s \geq 2$</td>
<td>120</td>
<td>210</td>
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In this case we have $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$.

Hence, by a direct computation using Kameko’s results for $k = 3$, one get

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<tbody>
<tr>
<td></td>
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<td>315</td>
</tr>
</tbody>
</table>

From the above results, we have
4. The case $\alpha(n + \mu(n)) = 3$

In this case we have $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$. By Main Theorem, for $s \geq 3$, $\dim(QP_4)_n = 15 \dim(QP_3)_m$ with $m = 2^{t+u} + 2^t - 2$.

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From the above results, we have

**Corollary.** Kameko’s conjecture is true for $k = 4$. 


